XLVI. *On the Reflexion and Refraction of Light.* *By* Sir William Thomson *.

1. GREEN'S doctrine † of incompressible elastic solid with equal rigidity, but unequal densities, on the two sides of an interface, to account for the reflexion and refraction of light, brings out, as is well known, for vibrations *perpendicular* to the plane of incidence (§ 12 below), exactly Fresnel's "sine-law:" and for vibrations *in* the plane of incidence a formula which agrees with Fresnel's tangent-law when the refractive index differs infinitely little from unity;—but which differs notably (enormously we may say) from it, and from the results of observation, in all practical cases:—in all cases, that is to say, in which the refractive index differs sufficiently from unity to have become subject of observation or measurement.

2. Since the first publication of Cauchy's work on the subject in 1830, and of Green's in 1837, many attempts have been made by many workers to find a dynamical foundation for Fresnel's laws of reflexion and refraction of light, but all hitherto ineffectually. On resuming my own efforts since the recent meeting of the British Association in Bath, I first ascertained that an inviscid fluid permeating among pores of an incompressible, but otherwise sponge-like, solid, does not diminish, but on the contrary augments, the deviation from Fresnel's law of reflexion for vibrations in the plane of incidence. Having thus, after a great variety of previous efforts which had been commenced in connexion with preparations for my Baltimore Lectures of this time four years ago, seemingly exhausted possibilities in respect to *incompressible* elastic solid, without losing faith either in light or in dynamics, and knowing that the condensational-rarefactional wave disqualifies ‡ any elastic solid of *positive* compressibility, I saw that nothing was left but a solid of such negative compressibility as should make the velocity of the condensational-rarefactional wave zero. So I tried it and immediately found that it, with other suppositions unaltered from Green's, exactly fulfils Fresnel's "tangent-law" for vibrations *in* the plane of incidence, and his "sine-law" for vibrations perpendicular to the plane of incidence. I then noticed that homogeneous air-less foam held from collapse by adhesion to a containing

* Communicated by the Author.

† Camb. Phil. Soc. Dec. 1837. Green's Collected Papers, pp. 246, 258, 267, 268.

‡ Green's Collected Papers, p. 246.

vessel, which may be infinitely distant all round, exactly fulfils the condition of zero velocity for the condensational-rarefactional wave; while it has a definite rigidity and elasticity of form, and a definite velocity of distortional wave, which can easily be calculated with a fair approximation to absolute accuracy.

3. Green, in his original paper "On the Reflexion and Refraction of Light," had pointed out that the condensational-rarefactional wave might be got quit of in two ways, (1) by its velocity being infinitely small, (2) by its velocity being infinitely great. But he curtly dismissed the former and adopted the latter, in the following statement:—"And it is not difficult to prove that the equilibrium of our medium would be unstable unless $A/B > 4/3$. We are therefore compelled to adopt the latter value of A/B *," (∞) "and thus to admit that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations, is very great compared with that of ordinary light." Thus originated the "jelly" theory of ether, which has held the field for fifty years against all dynamical assailants, and yet has failed to make good its own foundation.

4. But let us scrutinize Green's remark about instability. Every possible infinitesimal motion of the medium is, in the elementary dynamics of the subject, proved to be resolvable into coexistent condensational-rarefactional wave-motions. Surely, then, if there is a real finite propagational velocity for each of the two kinds of wave-motion, the equilibrium *must* be stable! And so I find Green's own formula † proves it to be *provided we either suppose the medium to extend all through boundless space, or give it a fixed containing vessel as its boundary.* If, left to itself in space, there be a bubble of air contained in the ordinary film, in which we suppose the tension to be constant however much it may expand or shrink, it will come to stable equilibrium in the form of a globe of such size that the pressure inwards on the air due to the tension of the film is equal to the air-pressure outwards. But if instead of being constant, the tension of the film varies as $t^{1-\kappa}$ (t denoting its thickness) the equilibrium will be stable‡ or unstable according as κ is positive or negative. A finite

* A and B are the velocities of the condensational and distortional waves respectively; suppose for a moment the density of the medium unity.

† Collected Papers, p. 253; formula (C).

‡ Provided instability of the film itself (by thicker parts having greater contractile force than thinner parts) is artificially guarded against by keeping it arbitrarily of uniform thickness.

portion of Green's homogeneous medium left to itself in space will have the same kind of stability or instability according as $A/B > 4/3$, or $A/B < 4/3$. In fact $A - \frac{4}{3}B$, in Green's notation, is what I have called the "bulk-modulus" of elasticity*, and denoted by k (being infinitesimal change of pressure divided by infinitesimal change from unit volume produced by it: or the reciprocal of what is commonly called "the compressibility"). B is what I have called the "rigidity," as an abbreviation for "rigidity-modulus," and which we may regard as essentially positive. Thus Green's limit $A/B > 4/3$ simply means positive compressibility, or positive bulk-modulus: and the kind of instability that deterred him from admitting any supposition of $A/B < 4/3$ is the spontaneous shrinkage of a finite portion if left to itself in a volume infinitesimally less, or spontaneous expansion if left to itself in a volume infinitesimally greater, than the volume for equilibrium. This instability is, in virtue of the rigidity of the medium, converted into stability by attaching the bounding surface of the medium to a rigid containing vessel. How much smaller than 4/3 may A/B be we now proceed to investigate, and we shall find, as we have anticipated, that for stability it is only necessary that A be positive.

5. Taking Green's formula (C), but to make the energy principle which it expresses clearer (he had not even the words "energy," or "work"!), let W denote the quantity of work required per unit volume of the substance, to bring it from its unstressed equilibrium to a condition of equilibrium in which the matter which was at (x, y, z) is at $(x+u, y+v, z+w)$, u, v, w being functions of x, y, z such that each of the nine differential coefficients du/dx, du/dy, ... dv/dx, ... &c. is an infinitely small numeric, we have

$$W = \tfrac{1}{2}\left\{ A\left[\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2\right.\right.$$
$$+B\left[\left(\frac{dw}{dy}+\frac{dv}{dz}\right)^2+\left(\frac{du}{dz}+\frac{dw}{dx}\right)^2+\left(\frac{dv}{dx}+\frac{du}{dy}\right)^2\right]$$
$$\left.-4B\left(\frac{dv}{dy}\frac{dw}{dz}+\frac{dw}{dz}\frac{du}{dx}+\frac{du}{dx}\frac{dv}{dy}\right)\right\} \quad . \quad . \quad . \quad (1).$$

This, except difference of notation, is the same as the formula

* 'Encyclopedia Britannica,' Article "Elasticity,": reproduced in Vol. III. of my Collected Papers, soon to be published.

for energy given in Thomson and Tait's 'Natural Philosophy,' § 693 (7).

6. To find the total work required to alter the given portion of solid from unstrained equilibrium to the strained condition (u, v, w) we must take $\iiint dx\,dy\,dz\,\mathrm{W}$ throughout the rigid containing vessel. Taking first the last line of (1); integrating the three terms each twice successively by parts in the well-known manner, subject to the condition $u=0$, $v=0$, $w=0$ at the boundary; we transform the factor within the last vinculum to

$$\iiint dx\,dy\,dz\left(\frac{dv}{dz}\frac{dw}{dy}+\frac{dw}{dx}\frac{du}{dz}+\frac{du}{dy}\frac{dv}{dx}\right).$$

Adding this with its factor $-4\mathrm{B}$ to the other terms of (1) under $\iiint dx\,dy\,dz$, we find finally

$$\mathrm{W}=\tfrac{1}{2}\iiint dx\,dy\,dz\left\{\mathrm{A}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)^2\right.$$
$$\left.+\mathrm{B}\left[\left(\frac{dw}{dy}-\frac{dv}{dz}\right)^2+\left(\frac{du}{dz}-\frac{dw}{dx}\right)^2+\left(\frac{dv}{dx}-\frac{du}{dy}\right)^2\right]\right\}\quad . \ (2).$$

This shows that positive work is needed to bring the solid to the condition (u, v, w) from its unstrained equilibrium and therefore its unstrained equilibrium is stable, if A and B are both positive, however small be either of them.

7. If $\mathrm{A}=0$, as we are going to suppose it for our optical problem, no work is required to give the medium any infinitely small irrotational displacement; and thus we see the explanation of the zero-velocity of the condensational and rarefactional wave which Green notices as corresponding to the case of $\mathrm{A}=0$. But for present convenience, and until the Aberration of Light, or, generally, the motion of ponderable bodies through ether and related questions of electrostatics, electric currents, and magnetism, come to be considered in connexion with conceivable qualities of the luminiferous ether, we shall suppose forces proportional to cubes of strains to act in such a manner as to render stable the equilibrium which is neutral or "labile" * with no other forces acting than those taken into account in (1) and (2) above. Accordingly, as a

* This word, very well chosen as it seems to me, has I believe been, by some French writers, employed to signify such equilibrium as that of a rigid body on a perfectly smooth horizontal plane, or of water in a rigid closed vessel entirely filled by it.

second approximation, correcting (1) and (2) in which only forces in simple proportion to strains are taken into account, we may now have, for the case of $A=0$,

$$W=\tfrac{1}{2}\iiint dx\,dy\,dz$$

$$\left\{B\left[\left(\frac{dw}{dy}-\frac{dv}{dz}\right)^2+\left(\frac{du}{dz}-\frac{dw}{dx}\right)^2+\left(\frac{dv}{dx}-\frac{du}{dy}\right)^2\right]+I_4\right\} \quad . \, (3),$$

where I_4 denotes an essentially positive homogeneous function of the fourth degree of $\frac{du}{dx}, \frac{du}{dy}, \ldots \frac{dv}{dx}, \ldots \frac{dw}{dy}, \frac{dw}{dz}$, isotropic relatively to (x, y, z).

8. The equations of motion of the general elastic solid, taken direct from the equations of equilibrium*, are, in terms of our previous notation adopted from Green, and with ζ to denote density,

$$\left.\begin{aligned}\zeta\frac{d^2u}{dt^2}&=\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz},\\ \zeta\frac{d^2v}{dt^2}&=\frac{dU}{dx}+\frac{dQ}{dy}+\frac{dS}{dz},\\ \zeta\frac{d^2w}{dt^2}&=\frac{dT}{dx}+\frac{dS}{dy}+\frac{dR}{dz},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4),$$

where u, v, w denote (as above from Green) displacements, not velocities ; P, Q, R, normal components of pull (per unit area) on interfaces respectively perpendicular to x, y, z ; and S, T, U respectively the tangential components of pull as follows :—

$$\left.\begin{array}{l} S\left\{\begin{array}{lllllllll} = & \text{pull} & \text{parallel} & \text{to } y & \text{on} & \text{face} & \text{perpendicular} & \text{to} & z, \\ = & " & " & z & " & " & " & & y, \end{array}\right. \\ T\left\{\begin{array}{lllllllll} = & & " & " & z & " & " & " & x, \\ = & & " & " & x & " & " & " & z, \end{array}\right. \\ U\left\{\begin{array}{lllllllll} = & & " & " & x & " & " & " & y, \\ = & & " & " & y & " & " & " & x. \end{array}\right. \end{array}\right\} \quad . \quad (5).$$

9. For an isotropic solid, using Green's notation A, B as above (§ 3), we have†

* Thomson and Tait, 'Natural Philosophy,' § 697.
† Thomson and Tait, 'Natural Philosophy,' § 693 and § 670 (6).

$$S=B\left(\frac{dw}{dy}+\frac{dv}{dz}\right);\quad T=B\left(\frac{du}{dz}+\frac{dw}{dx}\right),\quad U=B\left(\frac{dv}{dx}+\frac{du}{dy}\right)\ .\ (6);$$

$$P=A\delta-2B\left(\frac{dv}{dy}+\frac{dw}{dz}\right);\quad Q=A\delta-2B\left(\frac{dw}{dz}+\frac{du}{dx}\right);$$

$$R=A\delta-2B\left(\frac{du}{dx}+\frac{dv}{dy}\right)\ .\ .\ (7),$$

where

$$\delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\quad .\ .\ .\ .\ .\ .\ .\ (8).$$

Using these in (4) we find, as the equations of motion,

$$\zeta\frac{d^2u}{dt^2}=(A-B)\frac{d\delta}{dx}+B\nabla^2u;\quad \zeta\frac{d^2v}{dt^2}=(A-B)\frac{d\delta}{dy}+B\nabla^2v;$$

$$\zeta\frac{d^2w}{dt^2}=(A-B)\frac{d\delta}{dz}+B\nabla^2w\ .\ (9),$$

where

$$\delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\quad .\ .\ .\ .\ .\ .\ (10),$$

and

$$\nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\quad .\ .\ .\ .\ .\ (11).$$

10. From (9) we find, by taking d/dx of the first, d/dy of the second, and d/dz of the third, and adding,

$$\zeta\frac{d^2\delta}{dt^2}=A\nabla^2\delta\quad .\ .\ .\ .\ .\ .\ .\ (12).$$

Put now

$$u'=u-\frac{d}{dx}\nabla^{-2}\delta;\quad v'=v-\frac{d}{dy}\nabla^{-2}\delta;\quad w'=w-\frac{d}{dz}\nabla^{-2}\delta\ (13);$$

which implies

$$\frac{du'}{dx}+\frac{dv'}{dy}+\frac{dw'}{dz}=0\quad .\ .\ .\ .\ .\ .\ (14):$$

and we find, by (9),

$$\zeta\frac{d^2u'}{dt^2}=B\nabla^2u';\quad \zeta\frac{d^2v'}{dt^2}=B\nabla^2v';\quad \zeta\frac{d^2w'}{dt^2}=B\nabla^2w'\ .\ (15).$$

Equations (12), (14), and (15) prove the footnote to § 3: and they prove, further, that any infinitesimal disturbance whatever is composed of specimens of the condensational-rarefactional wave, and specimens of the distortional wave, coexisting: and, lastly, they prove that the displacement in the condensational-rarefactional wave is irrotational, because we see by (13) that an absolutely general expression for its components, $u-u'$, $v-v'$, $w-w'$, if denoted by u'', v'', w'', is

$$u''=\frac{d\Psi}{dx}, \quad v''=\frac{d\Psi}{dy}, \quad w''=\frac{d\Psi}{dz} \quad . \quad . \quad . \quad (16),$$

where Ψ is any function such that

$$\nabla^2\Psi=\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).$$

Hence, as δ satisfies (12), we have

$$\zeta\frac{d^2\Psi}{dt^2}=\mathrm{A}\nabla^2\Psi \quad . \quad . \quad . \quad . \quad . \quad (18);$$

and we see, finally, that the most general solution of the equations of infinitesimal motion is given by

$$u=u'+u'', \quad v=v'+v'', \quad w=w'+w'', \quad . \quad . \quad (19):$$

provided u', v', w' satisfy (14) and (15); and u'', v'', w'' satisfy (16) and (18).

11. Let us now work out the general problem of reflexion and refraction between two portions of homogeneous elastic solid sliplessly attached to one another at a plane interface, and having different densities, ζ, $\zeta_{/}$; different rigidities, B, $\mathrm{B}_{/}$; and different values, A, $\mathrm{A}_{/}$, for the condensational-rarefactional wave modulus. Thus, if α, β; $\alpha_{/}$, $\beta_{/}$ denote the velocities of the condensational-rarefactional, and of the distortional, waves respectively in the two mediums, we have

$$\left.\begin{array}{ll}\alpha=\sqrt{(\mathrm{A}/\zeta)}, & \alpha_{/}=\sqrt{(\mathrm{A}_{/}/\zeta_{/})} \\ \beta=\sqrt{(\mathrm{B}/\zeta)}, & \beta_{/}=\sqrt{(\mathrm{B}_{/}/\zeta_{/})}\end{array}\right\} \quad . \quad . \quad . \quad (20).$$

To avoid circumlocutions we shall suppose the interface horizontal, and call the two mediums, or solids, the upper and the lower respectively. Take OX vertically upwards; and OY horizontal to the right: and let the incident ray come from the left obliquely downwards, in the plane YOX; so that, if i denote the angle of incidence, the equation of the wave-front of the incident ray is

$$x\cos i+y\sin i=\text{const.}$$

12. Consider first the case of vibrations perpendicular to the plane of incidence. The medium being isotropic, no condensational waves can be generated at the interface. There is therefore just one reflected and one refracted ray; all vibrations are perpendicular to the plane of incidence; and all three waves are purely distortional. It is clear also that

the phases of all the three waves must agree at the interface. Thus if, to express the disturbance in the lower medium, we take

$$w = \sin(\omega t + l_{,}x + m_{,}y), \ (x \text{ negative}) \quad . \quad . \quad . \quad (21),$$

where

$$l_{,} = \cos i_{,}\,\omega/\beta_{,}, \quad m_{,} = \sin i_{,}\,\omega/\beta_{,} \quad . \quad . \quad (22),$$

$i_{,}$ denoting the angle of refraction; we must have, for the disturbance in the upper medium,

$$w = f\sin(\omega t + lx + my) + g\sin(\omega t - lx + my), \ (x \text{ pos.}) \quad (23),$$

where

$$l = \cos i\ \omega/\beta, \quad m = \sin i\ \omega/\beta \quad . \quad . \quad . \quad . \quad (24).$$

The agreement of phases all along the interface, that is for all values of y, requires

$$m = m_{,};$$

and therefore, by (22) and (24),

$$\sin i/\beta = \sin i_{,}/\beta_{,} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25),$$

which proves the "law of refraction." It, with (22) and (24), gives

$$l = m\cot i; \quad l_{,} = m\cot i_{,} \quad . \quad . \quad . \quad . \quad (26).$$

The other interfacial conditions are simply w continuous; and [§ 9, (6)] T, continuous; which give

$$f + g = 1; \ \text{ and } \ \mathrm{B}l(f-g) = \mathrm{B}_{,}l_{,} \quad . \quad . \quad . \quad (27);$$

whence

$$f = \tfrac{1}{2}\frac{\mathrm{B}l + \mathrm{B}_{,}l_{,}}{\mathrm{B}l}; \quad g = \tfrac{1}{2}\frac{\mathrm{B}l - \mathrm{B}_{,}l_{,}}{\mathrm{B}l} \quad . \quad . \quad . \quad (28);$$

$$\frac{g}{f} = \frac{\mathrm{B}l - \mathrm{B}_{,}l_{,}}{\mathrm{B}l + \mathrm{B}_{,}l_{,}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

In the case of equal rigidities, or $\mathrm{B} = \mathrm{B}_{,}$, this becomes

$$\frac{g}{f} = \frac{l - l_{,}}{l + l_{,}} = \frac{\cot i - \cot i_{,}}{\cot i + \cot i_{,}} = -\frac{\sin(i - i_{,})}{\sin(i + i_{,})} \quad . \quad (30),$$

which is Fresnel's "sine-law."

13. In the more difficult case of vibrations in the plane of incidence, we have two displacement-components, u, v, to consider, instead of only the one, w; and two surface-pull components, P and U, instead of the one, T: and our interfacial conditions now are u, v, P, U all continuous.

We have now condensational-rarefactional waves, besides distortional waves, to deal with: and it is therefore convenient to divide the solution according to (19); and, as a twodimensional solution of (14), to take

$$u'=\frac{d\phi}{dy},\quad v'=-\frac{d\phi}{dx}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

Thus we have

$$u=\frac{d\phi}{dy}+\frac{d\psi}{dx};\quad v=-\frac{d\phi}{dx}+\frac{d\psi}{dy}\quad . \quad . \quad . \quad (32);$$

which, by (6) and (7), give

$$\left.\begin{aligned} P&=2B\frac{d^2\phi}{dx\,dy}+\left(-2B\frac{d^2}{dy^2}+A\nabla^2\right)\psi;\\ U&=B\left[\left(\frac{d^2}{dy^2}-\frac{d^2}{dx^2}\right)\phi+2\frac{d^2\psi}{dx\,dy}\right]\end{aligned}\right\}\quad . \quad . \quad (33).$$

We may now, to represent the two refracted waves, assume, in the lower medium,

$$\phi=\sin(\omega t+l_{\prime}x+my)\ ;\quad \psi=C_{\prime}\sin(\omega t+\lambda_{\prime}x+my)\ . \quad (34);$$

and to represent the incident wave, supposed distortional, and the two reflected waves, in the upper medium

$$\phi=F\sin(\omega t+lx+my)+G\sin(\omega t-lx+my)\ ;$$
$$\psi=C\sin(\omega t-\lambda x+my)\quad . \quad (35),$$

where l, $l_{\prime}$, and m, still given by (22) and (24), verify (15), which give

$$\zeta\omega^2=B(l^2+m^2)\ ;\quad \text{and}\quad \zeta_{\prime}\omega^2=B_{\prime}(l_{\prime}^2+m^2)\quad . \quad (36);$$

and similarly, λ, $\lambda_{\prime}$, according to (18), are given by

$$\zeta\omega^2=A(\lambda^2+m^2);\quad \zeta_{\prime}\omega^2=A_{\prime}(\lambda_{\prime}^2+m^2)\quad . \quad . \quad (37).$$

Also, if by j and $j_{\prime}$ we denote the angles of reflexion and refraction of the condensational-rarefactional waves, we have, similarly to (25), (26),

$$\sin j/\alpha=\sin j_{\prime}/\alpha_{\prime}=\sin i/\beta\ ;\ \lambda=m\cot j\ ;\ \lambda_{\prime}=m\cot j_{\prime}\ . \quad (38).$$

14. The continuity of u, v, P, U, on the two sides of the interface, gives, by (32), (33), (34), (35);

$$m(F+G)-\lambda C=\ m+\lambda_{\prime}C_{\prime}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39);$$

$$-l(F-G)+mC=-l_{\prime}+mC_{\prime}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40);$$

$$-Blm(F-G)+[Bm^2-\tfrac{1}{2}A(\lambda^2+m^2)]C$$
$$=-B_{\prime}l_{\prime}m+[B_{\prime}m^2-\tfrac{1}{2}A_{\prime}(\lambda_{\prime}^2+m^2)]C_{\prime}\ . \quad (41);$$

$$B\{(l^2-m^2)(F+G)+2\lambda mC\}=B_{\prime}(l_{\prime}^2-m^2-2\lambda_{\prime}mC_{\prime})\qquad (42).$$

Eliminating $(F+G)$ from (39) and (42); and $(F-G)$ from (40), (41) we find two equations for C and $C_{\prime}$; and then (39), (40) give $(F+G)$, $(F-G)$; and thus we find our four unknown quantities. The resulting formulas are greatly simplified by the assumption of equal rigidities ($B=B_{\prime}$) adopted by Green on account of its simplicity, and proved by Lorentz and Rayleigh to be necessary, in the incompressible-solid theory, to get any approach to agreement with observation. It seems equally, or almost equally, necessary in the other extreme form of the elastic solid theory which I am now suggesting; but at all events I adopt it for the present on account of its simplicity. It gives, by the elimination of $(F-G)$ from (40) and (41),

$$A(\lambda^2+m^2)C=A_{\prime}(\lambda_{\prime}^2+m^2)C_{\prime} \quad . \quad . \quad . \quad . \quad (43),$$

or, by (37),

$$\zeta C=\zeta_{\prime}C_{\prime} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44):$$

whence, by elimination of $(F+G)$ from (39) and (42), and by (37),

$$\frac{\lambda C+\lambda_{\prime}C_{\prime}}{m}=\frac{l_{\prime}^2+l^2}{l^2+m^2}=\frac{\zeta_{\prime}-\zeta}{\zeta} \quad . \quad . \quad . \quad (45);$$

whence

$$C=m\frac{\zeta_{\prime}}{\zeta}\frac{\zeta_{\prime}-\zeta}{\zeta_{\prime}\lambda+\zeta\lambda_{\prime}}; \quad C_{\prime}=m\frac{\zeta_{\prime}-\zeta}{\zeta_{\prime}\lambda+\zeta\lambda_{\prime}} \quad . \quad . \quad . \quad (46).$$

This, used in (40), and (45) in (39), give

$$F-G=\frac{l_{\prime}}{l}+\frac{m^2(\zeta_{\prime}-\zeta)^2}{l(\zeta_{\prime}\lambda+\zeta\lambda_{\prime})\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47);$$

$$F+G=\frac{\zeta_{\prime}}{\zeta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

These (46), (47), (48), with (34),(35), and (32), express the complete solution of our problem.

15. To interpret it remark that (32) represent the components of three distinct waves in the upper medium, and two in the lower, of which the directions of propagation make angles with the normal to the interface equal respectively to $i, j, i_{\prime}, j_{\prime}$; [($j$, $i_{\prime}$, $j_{\prime}$ being given in terms of i by (25) and (38)]; and of which the amplitudes are as follows:—

$$\left.\begin{array}{lr}
\text{incident wave (distortional)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . & \omega\,.\,F/\beta \\
\text{distortional reflected wave} \quad . \quad . \quad . \quad . \quad . \quad . \quad . & \omega\,.\,G/\beta \\
\text{condensational-rarefactional reflected wave} \quad . & \omega\,.\,C/\alpha \\
\text{distortional refracted wave} \quad . \quad . \quad . \quad . \quad . \quad . \quad . & \omega\,.\,1/\beta_{\prime} \\
\text{condensational-rarefactional refracted wave} \quad . & \omega\,.\,C_{\prime}/\alpha_{\prime}
\end{array}\right\} (49).$$

16. To verify that the sum of the activities (rates of doing work per unit of time) of the four reflected and refracted waves is equal to the activity of the incident wave, consider pencils of them, all cutting the interface in a square with its sides respectively perpendicular and parallel to the plane of incidence. The activity of each of these pencils is equal to twice the kinetic energy in a length of it equal to its wave-length, divided by the common period; or, which is the same, twice its kinetic energy per unit volume, multiplied by its sectional area, multiplied by its propagational velocity. Now twice the kinetic energy per unit volume of a wave of either species in an elastic solid is equal to the density of the solid, multiplied by half the square of the maximum molar velocity; and the sectional areas of our five pencils are respectively $\cos i$, $\cos j$, $\cos i_{,}$, $\cos j_{,}$. Thus the activity of the incident pencil, for example, is

$$\zeta \cdot \tfrac{1}{2}\omega^4 (\mathrm{F}/\beta)^2 \cos i \beta \quad . \quad . \quad . \quad . \quad . \quad (50);$$

or, by (24),

$$\tfrac{1}{2}\omega^3 \zeta \mathrm{F}^2 l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51);$$

and is similarly found for the others. Hence the activities of the five pencils, each divided by $\frac{1}{2}\omega^3$, are respectively

$\zeta \mathrm{F}^2 l$ . . . incident (distortional) . . . . . (52);

$\zeta \mathrm{G}^2 l$ . . . distortional reflected . . . . . (53);

$\zeta_{,} l_{,}$ . . . distortional refracted . . . . . (54);

$\zeta \mathrm{C}^2 \lambda$ . . . condensational-rarefactional reflected (55);

$\zeta_{,} \mathrm{C}_{,}^2 \lambda_{,}$ . . . condensational-rarefactional refracted (56).

17. The first of these must be equal to the sum of the other four, and thus, subtracting from each side the second, we find, as a form of the equation of energies,

$$\zeta l (\mathrm{F}+\mathrm{G})(\mathrm{F}-\mathrm{G}) = \zeta_{,} l_{,} + \zeta \mathrm{C}^2 \lambda + \zeta_{,} \mathrm{C}_{,}^2 \lambda_{,} \quad . \quad . \quad (57),$$

which is verified by (47), (48), and (46). In verifying it we find, from (46),

$$\zeta \mathrm{C}^2 \lambda + \zeta_{,} \mathrm{C}_{,}^2 \lambda_{,} = m^2 \frac{\zeta_{,}}{\zeta} \frac{(\zeta_{,} - \zeta)^2}{\zeta_{,}\lambda + \zeta \lambda_{,}} \quad . \quad . \quad . \quad . \quad (58)$$

which is an important expression for the sum of the energies carried away per unit of time by the reflected and refracted condensational-rarefactional waves. In using these results, (52) . . . (58), it is convenient to remark that, by (24), (22), (38), we have

$$l = \frac{\omega}{\beta} \cos i; \quad m = \frac{\omega}{\beta} \sin i; \quad l_{,} = \frac{\omega}{\beta_{,}} \left(1 - \frac{\beta_{,}^2}{\beta^2} \sin^2 i\right)^{\frac{1}{2}} \quad . \quad (59);$$

$$\lambda=\frac{\omega}{\alpha}\left(1-\frac{\alpha^2}{\beta^2}\sin^2 i\right)^{\frac{1}{2}};\quad \lambda_{\prime}=\frac{\omega}{\alpha_{\prime}}\left(1-\frac{\alpha_{\prime}^2}{\beta^2}\sin^2 i\right)^{\frac{1}{2}} \quad . \quad . \quad (60);$$

and

$$\zeta/\zeta_{\prime}=\beta_{\prime}^2/\beta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

18. When α and $\alpha_{\prime}$ are small in comparison with β, we have, approximately,

$$\lambda=\omega/\alpha\ ;\quad \lambda_{\prime}=\omega/\alpha_{\prime};$$

and (58), and (52) with (47) (48), become approximately

$$\frac{\alpha}{\beta}\sin^2 i\ \frac{\zeta_{\prime}}{\zeta^2}\ \frac{(\zeta_{\prime}-\zeta)^2}{\zeta_{\prime}+\zeta\alpha/\alpha_{\prime}}\ \frac{\zeta\omega}{\beta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62),$$

and

$$\frac{1}{4}\sec i\left(\frac{\beta}{\beta_{\prime}}\cos i_{\prime}+\frac{\beta^2}{\beta_{\prime}^2}\cos i\right)^2\frac{\zeta\omega}{\beta} \quad . \quad . \quad (63);$$

which show that the energy carried away by the reflected and refracted condensational-rarefactional waves (62) is very small in comparison with the activity (63) of the incident distortional wave, whatever the angle of incidence. It is to be remarked that the wave-length of the condensational-rarefactional wave, in the upper medium for example, is α/β of the wave-length of the distortional wave, while, as we see by (61), (46), (47), (48), their amplitudes of vibration are comparable. Hence if we suppose α/β infinitely small, we must suppose the ratio of the vibrational amplitude of the incident wave to its wave-length to be infinitely small in comparison with α/β, in order that our formulas may still hold, or, which is the same, in order that the condensations and rarefactions may be infinitely small.

19. Without further preface, let $A=0$; which makes $\alpha=0$; and $\lambda=\infty$, and gives, by (47) and (48),

$$\frac{G}{F}=\frac{\dfrac{\zeta_{\prime}}{\zeta}-\dfrac{l_{\prime}}{l}}{\dfrac{\zeta_{\prime}}{\zeta}+\dfrac{l_{\prime}}{l}}=\frac{\dfrac{\sin^2 i}{\sin^2 i_{\prime}}-\dfrac{\cot i_{\prime}}{\cot i}}{\dfrac{\sin^2 i}{\sin^2 i_{\prime}}+\dfrac{\cot i_{\prime}}{\cot i}}=\frac{\tan(i-i_{\prime})}{\tan(i+i_{\prime})} \quad . \quad . \quad (64),$$

which is Fresnel's "tangent-formula."